Novelty Journals

ISSN 2394-7349

International Journal of Novel Research in Engineering and Science
Vol. 9, Issue 2, pp: (21-26), Month: September 2022 - February 2023, Available at: www.noveltyjournals.com

Fractional Power Series Method for Solving Fractional Differential Equations

Chii-Huei Yu

School of Mathematics and Statistics,

Zhaoqing University, Guangdong, China

DOI: https://doi.org/10.5281/zenodo.7153055

Published Date: 06-October-2022

Abstract: **Based on Jumarie type of Riemann-Liouville (R-L) fractional derivative, this paper provides some examples to illustrate how to use fractional power series to solve fractional differential equations. Chain rule and product rule for fractional derivatives and a new multiplication of fractional power series play important roles in this paper. In fact, our results are generalizations of the results of ordinary differential equations.**

Keywords: **Jumarie type of R-L fractional derivative, Fractional power series, Fractional differential equations, Chain rule, Product rule, New multiplication.**

I. INTRODUCTION

Fractional calculus is a mathematical tool used to study the derivatives and integrals of any order. It unifies and extends the concepts of derivative and integral of integer order. Generally speaking, many scientists do not know these fractional derivatives and integrals, nor do they use them in the purely mathematical field. However, in the past decades, fractional calculus has been widely used in many scientific fields, such as mechanics, electrical engineering, viscoelasticity, biology, physics, economics, etc [1-9].

The definition of fractional derivative is not unique. The commonly used definitions are Riemann-Liouvellie (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, conformable fractional derivative, and Jumarie's modified R-L fractional derivative [10-14]. Since Jumarie's modification of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie type of R-L fractional derivative and a new multiplication of fractional power series, some examples are given to illustrate how to use fractional power series method to solve fractional differential equations. Chain rule and product rule for fractional derivatives play important roles in this article. In fact, our results are generalization of those results of ordinary differential equations. On the other hand, the new multiplication we defined is a natural operation of fractional power series.

II. PRELIMINARIES

Firstly, we introduce the fractional calculus used in this paper.

Definition 2.1 ([15])**:** If $0 < \alpha \leq 1$, and x_0 is a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{1}$$

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where $\Gamma(\;)$ is the gamma function. Moreover, we define $\left({}_{x_0}D_x^{\alpha}\right)^n[f(x)] = \left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$, and it is called the n-th order α-fractional derivative of $f(x)$, where n is arbitrary positive integer.

Proposition 2.2 ([16])**:** *Let* α, β, x_0, c *be real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[c] = 0. \tag{3}$$

In the following, the definition of fractional power series is introduced.

Definition 2.3 ([17])**:** Assume that $0 < \alpha \leq 1$, x, x_0, and a_k be real numbers for all k. $\sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ is called a α-fractional power series at $x = x_0$.

Next, we introduce a new multiplication of fractional power series.

Definition 2.4 ([18])**:** If $0 < \alpha \leq 1$, and x_0 is a real number. Suppose that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}, \tag{4}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{5}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m} a_{k-m} b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{6}$$

In other words,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \otimes \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m} a_{k-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \end{aligned} \tag{7}$$

Definition 2.5 ([19])**:** Suppose that $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{8}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{9}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(g_\alpha(x^\alpha)\right)^{\otimes k}, \tag{10}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(f_\alpha(x^\alpha)\right)^{\otimes k}. \tag{11}$$

Definition 2.6 ([19])**:** Suppose that $0 < \alpha \leq 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \tag{12}$$

In addition, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2k}, \tag{13}$$

and

$$sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(2k+1)}. \tag{14}$$

Definition 2.7 ([20])**:** Let n be a positive integer, $\left(f_\alpha(x^\alpha)\right)^{\otimes n} = f_\alpha(x^\alpha) \otimes \cdots \otimes f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. Moreover, we define $\left(f_\alpha(x^\alpha)\right)^{\otimes 0} = 1$.

Theorem 2.8 (chain rule for fractional derivatives) ([21])**:** *If* $0 < \alpha \leq 1$, *and assume that* $f_\alpha(x^\alpha), g_\alpha(x^\alpha)$ *are* α*-fractional power series at* $x = x_0$*. Then*

$$\left({}_{x_0}D_x^\alpha\right)\left[f_\alpha\left(g_\alpha(x^\alpha)\right)\right] = \left({}_{x_0}D_x^\alpha\right)[f_\alpha(x^\alpha)]\left(g_\alpha(x^\alpha)\right) \otimes \left({}_{x_0}D_x^\alpha\right)[g_\alpha(x^\alpha)]. \tag{15}$$

Theorem 2.9 (product rule for fractional derivatives) ([21])**:** *Let* $0 < \alpha \leq 1$, *and* $f_\alpha(x^\alpha)$ *and* $g_\alpha(x^\alpha)$ *be* α*-fractional power series at* $x = x_0$, *then*

$$\left({}_{x_0}D_x^\alpha\right)[f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha)] = \left({}_{x_0}D_x^\alpha\right)[f_\alpha(x^\alpha)] \otimes g_\alpha(x^\alpha) + f_\alpha(x^\alpha) \otimes \left({}_{x_0}D_x^\alpha\right)[g_\alpha(x^\alpha)]. \tag{16}$$

III. EXAMPLES

In this section, we give some examples to illustrate how to use fractional power series method to solve fractional differential equations.

Example 3.1: Let $0 < \alpha \leq 1$. Solve the initial problem of first order α-fractional differential equation

$$\left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)] - \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2} - E_\alpha(y_\alpha) = 0, \tag{17}$$

$$y_\alpha(0) = 0. \tag{18}$$

Solution Let the solution be

$$y_\alpha(x^\alpha) = y_\alpha(0) + \frac{\left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)](0)}{\Gamma(\alpha+1)}x^\alpha + \frac{\left({}_0D_x^\alpha\right)^2[y_\alpha(x^\alpha)](0)}{\Gamma(2\alpha+1)}x^{2\alpha} + \frac{\left({}_0D_x^\alpha\right)^3[y_\alpha(x^\alpha)](0)}{\Gamma(3\alpha+1)}x^{3\alpha} + \frac{\left({}_0D_x^\alpha\right)^4[y_\alpha(x^\alpha)](0)}{\Gamma(4\alpha+1)}x^{4\alpha} + \cdots. \tag{19}$$

Then by this fractional differential equation and chain rule and product rule for fractional derivatives,

$$\left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)] = \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2} + E_\alpha(y_\alpha), \tag{20}$$

$$\left({}_0D_x^\alpha\right)^2[y_\alpha(x^\alpha)] = 2 \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha + E_\alpha(y_\alpha) \otimes \left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)], \tag{21}$$

$$\left({}_0D_x^\alpha\right)^3[y_\alpha(x^\alpha)] = 2 + E_\alpha(y_\alpha) \otimes \left({}_0D_x^\alpha\right)^2[y_\alpha(x^\alpha)] + E_\alpha(y_\alpha) \otimes \left(\left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)]\right)^{\otimes 2}, \tag{22}$$

$$\begin{aligned}\left({}_0D_x^\alpha\right)^4[y_\alpha(x^\alpha)] = {} & E_\alpha(y_\alpha) \otimes \left({}_0D_x^\alpha\right)^3[y_\alpha(x^\alpha)] + 3\, E_\alpha(y_\alpha) \otimes \left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)] \otimes \left({}_0D_x^\alpha\right)^2[y_\alpha(x^\alpha)] \\ & + E_\alpha(y_\alpha) \otimes \left(\left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)]\right)^{\otimes 3}.\end{aligned} \tag{23}$$

Takig $x = 0$ and use the initial condition $y_\alpha(0) = 0$, we obtain

$$\left({}_0D_x^{\alpha}\right)[y_{\alpha}(x^{\alpha})](0)=1, \tag{24}$$

$$\left({}_0D_x^{\alpha}\right)^2[y_{\alpha}(x^{\alpha})](0)=1, \tag{25}$$

$$\left({}_0D_x^{\alpha}\right)^3[y_{\alpha}(x^{\alpha})](0)=4, \tag{26}$$

$$\left({}_0D_x^{\alpha}\right)^4[y_{\alpha}(x^{\alpha})](0)=8. \tag{27}$$

So we get the fractional power series solution of this α-fractional differential equation

$$y_{\alpha}(x^{\alpha})=\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}+\frac{4}{\Gamma(3\alpha+1)}x^{3\alpha}+\frac{8}{\Gamma(4\alpha+1)}x^{4\alpha}+\cdots. \tag{28}$$

Example 3.2: Suppose that $0<\alpha\leq 1$. Find the general solution of first order α-fractional differential equation

$$\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes\left({}_0D_x^{\alpha}\right)[y_{\alpha}(x^{\alpha})]-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+2\right)\otimes y_{\alpha}(x^{\alpha})=-2\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}-2\frac{1}{\Gamma(\alpha+1)}x^{\alpha}. \tag{29}$$

Solution Let the general solution be

$$y_{\alpha}(x^{\alpha})=\sum_{k=0}^{\infty}a_k\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes k}. \tag{30}$$

Then by chain rule for fractional derivatives,

$$\left({}_0D_x^{\alpha}\right)[y_{\alpha}(x^{\alpha})]=\sum_{k=1}^{\infty}ka_k\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes(k-1)}, \tag{31}$$

Taking into the original equation to get

$$0=\sum_{k=1}^{\infty}ka_k\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes k}-\sum_{k=0}^{\infty}a_k\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes(k+1)}-2\sum_{k=0}^{\infty}a_k\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes k}+2\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}+2\frac{1}{\Gamma(\alpha+1)}x^{\alpha}$$

$$=-2a_0+(-a_1-a_0+2)\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+(-a_1+2)\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}+\sum_{k=3}^{\infty}\left((k-2)a_k-a_{k-1}\right)\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes k}. \tag{32}$$

Thus,

$$a_0=0,\ a_1=2,\text{ and }a_k=\frac{1}{k-2}a_{k-1}\text{ for all }k\geq 3. \tag{33}$$

Hence,

$$a_3=a_2,\ a_4=\frac{1}{2}a_3=\frac{1}{2!}a_2,\ a_5=\frac{1}{3}a_4=\frac{1}{3!}a_2,\ a_6=\frac{1}{4}a_5=\frac{1}{4!}a_2,\cdots \tag{34}$$

Therefore, the general solution of this α-fractional differential equation is

$$y_{\alpha}(x^{\alpha})=2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+a_2\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\otimes\left[1+\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+\frac{1}{2!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}+\frac{1}{3!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 3}+\cdots\right]$$

$$=2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+a_2\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\otimes E_{\alpha}(x^{\alpha}). \tag{35}$$

Where a_2 is any constant.

Example 3.3: If $0<\alpha\leq 1$. Solve initial value problem of second order α-fractional differential equation

$$\left({}_0D_x^{\alpha}\right)^2[y_{\alpha}(x^{\alpha})]+y_{\alpha}(x^{\alpha})=0. \tag{36}$$

$$y_{\alpha}(0)=-5,\ \left({}_0D_x^{\alpha}\right)[y_{\alpha}(x^{\alpha})](0)=3. \tag{37}$$

Solution Let the solution be

$$y_{\alpha}(x^{\alpha})=\sum_{k=0}^{\infty}a_k\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes k}. \tag{38}$$

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Vol. 9, Issue 2, pp: (21-26), Month: September 2022 - February 2023, Available at: www.noveltyjournals.com

Using chain rule for fractional derivatives yields

$$\left({}_0D_x^{\alpha}\right)[y_{\alpha}(x^{\alpha})]=\sum_{k=1}^{\infty}ka_k\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes(k-1)}, \tag{39}$$

and hence,

$$\left({}_0D_x^{\alpha}\right)^2[y_{\alpha}(x^{\alpha})]=\sum_{k=2}^{\infty}k(k-1)a_k\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes(k-2)}. \tag{40}$$

Taking into the original equation to obtain

$$0=\sum_{k=2}^{\infty}k(k-1)a_k\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes(k-2)}+\sum_{k=0}^{\infty}a_k\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes k}$$

$$=(a_0+2a_2)+(a_1+6a_3)\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+(a_2+12a_4)\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}+(a_3+20a_5)\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 3}+\cdots. \tag{41}$$

Hence,

$$a_0+2a_2=0,\ a_1+6a_3=0,\ a_2+12a_4=0,\ a_3+20a_5=0,\cdots \tag{42}$$

Thus,

$$a_2=-\frac{1}{2!}a_0,\ a_3=-\frac{1}{3!}a_1,\ a_4=\frac{1}{4!}a_0,\ a_5=\frac{1}{5!}a_1,\ a_6=-\frac{1}{6!}a_0,\cdots \tag{43}$$

So, the general solution of this α-fractional differential equation is

$$y_{\alpha}(x^{\alpha})=a_1\left[\frac{1}{1!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)-\frac{1}{3!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 3}+\frac{1}{5!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 5}-\cdots\right]+a_0\left[1-\frac{1}{2!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}+\frac{1}{4!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{4}-\cdots\right]$$

$$=a_1\sin_{\alpha}(x^{\alpha})+a_0\cos_{\alpha}(x^{\alpha}). \tag{44}$$

Where a_1, a_0 are any constants.

Since $y_{\alpha}(0)=-5$, $\left({}_0D_x^{\alpha}\right)[y_{\alpha}(x^{\alpha})](0)=3$. It follows that $a_0=-5$, $a_1=3$. Consequently, the fractional power series solution of this initial value problem is

$$y_{\alpha}(x^{\alpha})=3\sin_{\alpha}(x^{\alpha})-5\cos_{\alpha}(x^{\alpha}). \tag{45}$$

IV. CONCLUSION

In this paper, we provide three examples to illustrate how to use fractional power series to solve fractional differential equations based on Jumarie's modified R-L fractional derivative. A new multiplication of fractional power series, and chain rule and product rule for fractional derivatives play important roles in this article. In fact, the new multiplication we defined is a natural operation of fractional power series. In the future, we will use Jumarie type of R-L fractional derivative and fractional power series method to expand the research field to engineering mathematics and fractional calculus problems.

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